BRX TH-6308
CALT-TH 2016-026

One-loop gravity divergences in $D > 4$ cannot all be removed

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Abstract

Unlike the one-loop QG divergences in $D = 4$, which can all be transformed away, those in arbitrary higher (even) D cannot.

1 $D = 4$

Explicit loop calculations in quantum gravity (QG) originated over four decades ago with 'tHooft and Veltman's [1] one-loop results in 4D. They noted that, even without calculation, the one-loop, hence quadratic in curvature, leading correction of the source-free theory could always be removed: First, the Riemann-squared term can be turned into a sum of Ricci- and scalar- curvature-squared ones using the pure divergence nature of the 4D Gauss-Bonnet (GB) invariant,

$$\int d^4x \epsilon^{\alpha\beta\gamma\delta}\, \epsilon^{\rho\sigma\mu\nu}\, R_{\alpha\beta\rho\sigma}\, R_{\gamma\delta\mu\nu} \sim \int d^4x [R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} - 4\, R_{\mu\nu}\, R^{\mu\nu} + R^2] \sim 0. \tag{1}$$

Second, the remaining, $\int R^2_{\mu\nu}$ and $\int R^2$, terms can then be relegated to higher order by the field redefinition

$$\delta g_{\mu\nu} = a(g, R)\, R_{\mu\nu} + b(g, R)\, g_{\mu\nu} R, \tag{2}$$

one that is clearly available in all D to remove monomials involving at least one Ricci tensor or scalar. [It is essential that the quadratic corrections are, by definition, NOT to be taken as part of the "kinetic" term, which remains R, and of course cannot be removed.]

The development of string theory and its corrections to QG in $D = 10$ has made the study of higher D relevant (see [2] for a recent example). We note here, in the same no-calculation spirit, that the no-correction result breaks down in $D = 2n > 4$, because there is now more than one independent monomial in $(R_{\mu\nu\alpha\beta})^n$, while there is still only one, GB ($D > 4$), available to remove them. Even-parity counterterms (and GB) cannot exist at all in odd dimensions, since they would have to have an odd number of derivatives, hence of indices: Power counting in GR trivially establishes that all one-loop counterterms are of D-derivative order because vertices/propagators are universally of $+2/-2$ derivative order and cancel each other when adding a new vertex, so that the cutoff dependence is just that of the integration dimension, $\int d^D p$, for any number of external gravitons.

2 $D > 4$

The relevant GB invariant in any even, $D = 2n$, (1) simply generalizes to

$$GB(D = 2n) = \int d^{2n}x \epsilon^{\cdots} \epsilon^{\cdots} (R^1_{\cdots} \;\; R^n_{\cdots}) \sim 0, \tag{3}$$

while of course (2) still removes all terms involving at least one Ricci or curvature scalar in any D. Consider $D = 6$ for concreteness; there, counterterms and GB are of 6-derivative order. The candidate one-loop invariants are then of three types,

$$\int \mathrm{Riem}^3, \qquad \int \mathrm{Riem}\, d\, d\, \mathrm{Riem}, \qquad GB(6), \tag{4}$$

(d denotes a covariant derivative), plus (harmless) terms containing at least one Ricci or R factor. The single GB can only convert one of the two independent Riem^3 into the "Ricci" type. The $\int \mathrm{Riem}\, d\, d\mathrm{Riem}$ can be reduced to Riem^3 form by the cyclic Bianchi identities $R_{\mu\nu[\alpha\beta;\rho]} = 0$, $\mathrm{div}\, \mathrm{Riem} = \mathrm{curl}\, \mathrm{Ricci}$, to remove the dd in favor of a possible extra Riem where use of $[d, d] \sim \mathrm{Riem}$ is needed. The same conclusions apply, *a fortiori*, to any higher D; a detailed list of curvature monomials for any D can be found in [3]. A separate interesting set of actions, the Weyl invariant ones, also increases with dimension, beyond the unique $\int C^2$ of $4D$; a sample is given in [4], there being two $\int C^3$ (and one $\int C\Box C$) in 6D: there are now multiple Weyl "gravities".

3 Comments

We close with some conceivable, if unlikely, possible exceptions to these no-calculation statements. First, although there is no obvious selection principle that would allow only the one GB $(\mathrm{Riem})^n$ counterterm (3) from among the many possible ones at higher dimensions, this should, and can, easily be checked explicitly by using the methods of [1]. Second, the catalog of [3] lists the locally independent monomials; there might be some integral relations among them – that is, under the integral sign, they may differ from each other by divergences and terms involving Riccis. This too seems unlikely (except for the $\int \mathrm{Riem}^p \Box^q \mathrm{Riem}^{n-p-q}$ type above), simply because 4D has the special property that, to leading, $\sim h^2_{\mu\nu}$, order, all momenta are sandwiched between just two h, so on-Einstein shell where $0 = \mathrm{Ricci} = (pph)_{\mu\nu}$, there is only one $(hp^4h) \sim (\mathrm{Riem})^2$ monomial. At higher D, the leading terms are now of the form $hpp \ldots hpp \ldots h \ldots$; the p and h indices, are more "segregated", which accounts for the increase of $\int \mathrm{Riem}^n$ combinations.

Acknowledgements.

I thank J. Franklin for Latexerity. This work was supported by grants NSF PHY-1266107 and DOE # desc0011632.

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